FINITE ELEMENT TECHNIQUES FOR THE SOLUTION OF POISSON'S EQUATION

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Summary

This paper describes a number of improvements to the finite-element method. The functional, whose extremum is furnished by the solution of Poisson's equation over the union of a number of piecewise homogeneous regions, is presented. The Rayleigh-Ritz method, using a two variable power series as a trial function, is employed to find an approximation to the solution. It is shown that Cauchy and Neumann boundary conditions are natural ones for the functional and that the interface condition of continuity of normal flux is satisfied naturally as well. The method of formulating the Dirichlet boundary condition, as a natural one, is described. The paper shows that a curved boundary need not be approximated by triangle sides but may be defined as accurately as desired.

Functionals and Boundary Conditions

The paper describes the application of the variational method to problems involving the Cauchy boundary condition

$$\left.\frac{\partial\phi}{\partial n}\right|_s + \sigma(s)\ \phi(s) = h(s) \qquad (1)$$

of which the Neumann boundary condition

$$\left.\frac{\partial\phi}{\partial n}\right|_s = h(s) \qquad (2)$$

is a special case. The Cauchy condition caters for the case in which, say, current flow across a boundary is proportional to the potential difference across it. The inhomogeneous Neumann condition represents a source distribution along the doundary. The homogeneous Neumann condition, i.e. (2) with $h(s) = 0$, corresponds, say, to an insulating boundary.

The relevant functional for the solution of Poisson's equation

$$\nabla^2\phi = f(x,y) \qquad (3)$$

where $f(x,y)$ is a known function, is

$$F(u) = \iint_R [(\nabla u)^2 - 2fu]\,dxdy + \oint [\sigma(s)u^2 - 2h(s)u]\,ds \qquad (4)$$

$u(x,y)$ is a trial function that is chosen in an attempt to minimize $F(u)$. It will be shown that (1) and (2) are natural boundary conditions of the functional (4).

Morse and Feshbach[1] indicate that by addition of the integral $-2\oint \frac{\partial u}{\partial n}\,u\,ds$ to the functional for the Helmholtz equation, the homogeneous Dirichlet condition is made natural. The paper will show that the Dirichlet boundary condition

$$\phi(s) = g(s) \qquad (5)$$

can be made natural by altering (4) to[2]

$$F(u) = \iint_R [(\nabla u)^2 - 2fu]\,dxdy + \int_{C_1} [\sigma(s)u^2 - 2h(s)u]\,ds + 2\int_{C_2} [g(s) - u(s)]\,\frac{\partial u}{\partial n}\,ds \qquad (6)$$

The contour integrals C_1 and C_2 pertain to those regions of the boundary over which the Cauchy and Dirichlet conditions, respectively, hold.

Curved Boundaries and Interfaces

Results will be presented for arbitrary boundary shapes. Previous applications of finite element methods have used the sides of triangles to approximate boundaries in a polygonal sense. Due to severe limitations on the number of triangles that can be accomodated, curved boundaries cannot be accurately represented. In the work to be reported, two vertices of each triangle adjacent to a curved boundary are located on the boundary. The boundary may curve into the triangle or away from it. If the former, that part of the surface integral (4), relating to the triangle being considered, is performed not over the entire triangle but only over that part of it in the interior of the region. If the boundary curves away from the triangle side, then the integration is performed over the extra adjacent area using the polynomial defined within the triangle. Similarly, the contour integral follows the boundary exactly.

To perform these integrations, the boundary may be defined in a piecewise-polynomial fashion. The resulting integrations all involve polynomials and may be accomplished fairly easily. In the work reported, the boundary was defined in a piecewise-linear (polygonal) sense. This should not be confused with the approximation of a boundary by triangle sides. The latter approach is limited as to the number of linear segments available whereas the former permits one to use virtually as many as desired in order to approximate the boundary as closely as desired.

Finally, results will be presented for the case of a piecewise homogeneous medium in which the interfaces can assume any shape. It will be shown that the interface condition

$$\varepsilon_1 \left. \frac{\partial \phi_1}{\partial n} \right|_s = \varepsilon_2 \left. \frac{\partial \phi_2}{\partial n} \right|_s \tag{7}$$

is natural as long as the trial functions are continuous everywhere. To ignore this constraint will introduce unnecessary errors. As a special case, with $\varepsilon_1 = \varepsilon_2$, it is seen that if continuity of potential is enforced across subregions, as is usually done in the finite-element method, we are guaranteed that continuity of normal derivative is a natural condition.

References

1. P.M. Morse and H. Feshbach, Methods of Theoretical Physics, part II. New York: McGraw-Hill, 1953, pp. 1131-1133.

2. T.Hazel and A. Wexler, "Variational formulation of the Dirichlet boundary condition." (To be published).

Notes